Binghamton University
The Open Repository @ Binghamton (The ORB)

The Society for Ancient Greek Philosophy Newsletter

3-28-2003

Plato's Geometrical Logic

Mark Faller
Alaska Pacific University, markfaller@yahoo.com

Follow this and additional works at: https://orb.binghamton.edu/sagp

Part of the Ancient History, Greek and Roman through Late Antiquity Commons, Ancient Philosophy Commons, and the History of Philosophy Commons

Recommended Citation
Faller, Mark, "Plato's Geometrical Logic" (2003). *The Society for Ancient Greek Philosophy Newsletter*. 339.
https://orb.binghamton.edu/sagp/339

This Article is brought to you for free and open access by The Open Repository @ Binghamton (The ORB). It has been accepted for inclusion in The Society for Ancient Greek Philosophy Newsletter by an authorized administrator of The Open Repository @ Binghamton (The ORB). For more information, please contact ORB@binghamton.edu.

Plato's Geometrical Logic

Mark Faller

Presented to the Society for Ancient Greek Philosophy at its meeting with the Pacific Division of the American Philosophical Association, March 28, 2003, San Francisco

Socrates' brief mention of a complex problem in geometrical analysis at *Meno* (86d-87c) remains today a persistent mystery. The ostensible reason for the reference is to provide an analogy for the method of hypothesis from the use of hypotheses in analytic geometry. Both methods begin by assuming what is to be demonstrated and then show that the assumption leads to a well-founded truth rather than something known to be false.

But why did Plato pick this particular problem in analysis and why at this particular place in the inquiry? For those of us who view the dialogues as pedagogical puzzles for readers of all time to "scour" out the subtle and complicated details, this is an unquiet mystery that demands further examination.

In this paper I will defend the claim that Plato had developed a powerful new heuristic method for the clarification and resolution of a broad range of philosophical problems. This method, based on the techniques of inquiry used in geometry, was a kind of conceptual analysis using geometry as an interpretive model to elaborate the allowable structures of logic. I will argue that Plato adopted geometrical constructions to navigate the precarious passage between the syntactical and semantic structures in philosophical arguments, and that this procedure amounts to a kind of logic of discovery.

My approach in this inquiry will be twofold. I will first attempt to elaborate the precise philosophical question that the method of analysis is meant to resolve. I will pay particular attention to any critical hurdles that remain intractable within the limits of the conversation itself. I will then look closely at the details of the geometrical problem to determine just how it could possibly reflect any light back onto the original inquiry.

There have been many other interesting attempts to figure out just what Plato meant by "analysis" or the method of hypothesis[1]. They have all focused completely on the interpretation textual references in the dialogues and the commentaries. And they have achieved almost no common ground of assent as to the nature of this process. By examining at close quarters the specific problem in geometrical analysis that Plato wished to use as his example, I hope to overcome some of the contention that originates from the ambiguity of relying merely on the translation of terms.

The Philosophical Problem

In the second half of the dialogue Socrates and Meno examine two independent and seemingly contradictory arguments on whether virtue is teachable. The first argument (86c-89c) attempts to demonstrate that virtue is teachable since:

If virtue is knowledge then it is teachable
Virtue is knowledge
Virtue is teachable

Socrates, in a side argument had established that since virtue is "good" and that "there is nothing good that knowledge does not encompass (87d)," that virtue is a kind of knowledge:

---

[1] Francis Cornford, "Mathematics and Dialectic in the *Republic VI-VII. (I.)*, *Mind* 41: 37-52, Richard Robinson, "Analysis in Greek Geometry", *Mind* 45: 464-73, Norman Gulley, "Greek Geometrical Analysis", *Phronesis*, 3, (1958), Patrick Byrne, *Analysis and Science in Aristotle*, (Albany, 1997), Jaakko Hintikka, and Unto Remes, *The Method of Analysis*, (Boston, 1974), Michael Mahoney, "Another Look at Greek Geometrical Analysis," *Archives for the History of Exact Sciences*, 5:319-48, Richard Robinson, *Plato's Earlier Dialectic*, (Oxford, 1953), Kenneth Dorter, *Form and Good in Plato's Eleatic Dialogues*, (Berkeley), Charles Kahn, *Plato's Socratic Dialogues*, (Cambridge, 1996).

S: Virtue then, as a whole or in part, is wisdom?
M: What you say, Socrates, seems to me quite right.

The result of this first argument, as arrived at by Meno, seems to be a reasonable solution to his own original question: "Necessarily, as I now think, Socrates, and clearly, on our hypothesis, if virtue is knowledge, it can be taught (89c)." Meno would not have challenged this result further and would have been satisfied to memorize the argument for passage back to Gorgias.

Socrates seems less tolerant of his own offspring: "Perhaps, by Zeus, but may it be that we were not right to agree to this (89c)?" He initiates a second argument with the examination of a further hypothesis, that "if virtue is teachable, then there would be teachers of it." After an examination of the empirical issue of whether there are in fact any teachers of virtue, the conversants arrive at the inevitable conclusion:

If virtue is teachable then there will be teachers of virtue
There are no teachers of virtue
Virtue is not teachable

This second result is both troubling and ironic. It is troubling for its inconsistency with the result of our earlier argument. We have proved that virtue both is and is not teachable.

The irony arises from the way in which these two results are related. Socrates wants to pursue the question "what is virtue". Meno only wants to know whether it is teachable. In the first argument Socrates leads Meno to the conclusion that virtue is teachable from the acknowledgment that virtue is a kind of knowledge. Socrates challenges this result on the integrity that is does not follow the method of hypothesis. The method of hypothesis is utilized precisely to examine the properties of a subject whose nature has not been explicitly laid out.

If we are going to resolve the opposing conclusions of these two arguments, we will need to examine more closely the precise nature of this method of hypothesis. We will also need to determine how the method of hypothesis is in any way illuminated by the procedures of geometrical analysis.

The second half of the *Meno* - the part which is explicitly stated to be working with the method of hypothesis as an example of the analysis of the mathematicians - includes a series of hypotheses meant to answer the question whether or not virtue can be taught. There are two significant features of the method of the geometers that point us towards their relevance for our philosophical problems in the *Meno*.

First, the hypotheses are structured and organized to follow the *diorismic* orientation of the mathematical example of analysis. A *diorism* is a form of geometrical reasoning that seeks to find the conditions for the possibility of a construction. Since constructions are of the nature of an unelaborated hypothesis, the purpose of the philosophical *diorism* will be to work out the conditions for the possibility of an assumed hypothesis.

Second, the hypotheses refer to a *reduction* of one problem to another with a more tractable solution. In the geometrical problem we are to determine whether a certain rectilinear figure could be constructed along the diameter of a circle by examining the hypothesis: "If this triangle is such that, when laid along the given line [as a rectangle of equal area] it is short by a space as large as the figure laid along it, then I think that one thing follows, whereas another thing follows if this cannot be done. So I am willing to tell you on this hypothesis, about inscribing the triangle, whether it is impossible or not"(87b).

Socrates then suggests that they should do the same with virtue: "let us investigate whether it is teachable or not by means of a hypothesis, and say this: Among the things existing in the soul, of what sort is virtue, that it should be teachable or not? First, if it is another sort than knowledge, is it teachable or not, or, as we were saying, recollectable"(87b)? As with the geometrical problem we are going to examine the hypothesis of whether virtue is teachable by considering some "reduced" set of conditions: If it meets the condition, then one thing follows, if not, then another.

As in the mathematical problem, the philosophical "contrarial pair" is set up as a mutually exclusive set of conditionals. The first fork is put forward as a strong assertion by Socrates: "Isn't it plain to everyone that a man is not taught anything except knowledge (87c)?" This statement can be logically restructured as " If not knowledge, then not teachable."

The second fork, which Socrates offers up almost immediately, is the more recognizable conditional: "If virtue is knowledge then it is teachable (87c)." The two conditionals together seem to make up the kind of pair needed to meet the conditions of our *diorism*: If virtue is knowledge then it is teachable, and if virtue is not knowledge then it is not teachable.

We should take note of the logical structure of these two Contrarial Pairs. They are both (the geometrical and the one about the teachability of virtue) set up as exclusive and exhaustive contraries, or as not admitting any middle. This puts both sets of hypotheses on the same footing as Meno's paradox. There is only the possibility of teachability and knowledge or non-teachability and not knowledge. There can be no third alternative.

But the Slave Boy problem (Double Square) raised the possibility of there being such a middle ground. The unknown had been seemingly learned. The examination of the hypothesis of whether virtue is teachable will be equally an examination of the conditions whereby this middle ground between *contrary* propositions may be established. Socrates will need to establish some kind of propositional "middle" between these contraries, corresponding to the "mean" of the Double Square problem.

This seeking "middles" also returns us to one of the original meanings of analysis: "It follows, then, that in all our inquiries, we inquire either (a) whether there is a middle or (b) what the middle is, for the cause is the middle, and in all cases it is this that is sought (*Post. Anal. B, 90a,5*). In this sense the Slave Boy or the Double Square problem is a kind of analysis since it sought to find a "middle": the mean proportional between the sides of the given square.

There are some features we should note about the setting up of this pair of conditionals. First we must evaluate whether or not the conditionals are really exhaustively exclusive. It is clear that one of the possibilities that this pair excludes is that there might be some knowledge that is not teachable. That this possibility is excluded becomes immediately evident in Socrates' rephrasing of the pair in terms of the contrapositive of the negative conditional - If virtue is not knowledge then it is not teachable = If teachable then knowledge:

S: We thought it could be taught if it was knowledge? - Yes.
S: And that it was knowledge, if it could be taught? - Quite so (98d).

This restructuring of the hypotheses makes it clearer that the condition for the exclusivity of the original two conditionals is that they together, as restructured, make up such a biconditional. Or, in other words, teaching and knowledge must be interchangeable if the conditionals are to be convertible with one another. So that if we have a convertible syllogism (i.e. one with a biconditional major premise), then there *will not* be a kind of knowledge that is unteachable, for the contrarial pair will be mutually exclusive (If teachable then knowledge and if not teachable then not knowledge).

We have seen in the *Meno*, Plato is working with what approximates to a traditional approach to the convertibility of arguments. The condition for the convertibility of an analysis into a synthesis - the exchanging of the minor premise and the conclusion - is the biconditionality, or conversion of the major premise:

Analysis	**Synthesis**
All P is M	All M is P (Convertible Major Premise)
All S is P (Hypothesis)	All S is M
All S is M (Show to be true)	All S is P (Demonstrated Hypothesis)

In analysis we assume what we wish to prove; for example, "All S is P" in the syllogism on the left. If we can show that the assumed hypothesis leads to a true conclusion, we can then conclude that the corresponding synthetic syllogism is valid: i.e. that the major premise converts (is a biconditional).

It should be clarified at this point that we have been referring to convertibility in two fundamentally distinct senses. Individual propositions may convert and arguments may also be said to convert. With propositions, conversion is just the exchanging of the subject and predicate. This kind of exchange is always valid with two kinds of propositions: particular positive statements (Some S are P/ Some P are S) and negative universal statements (No S are P/ No P are S). In order for positive universally categorical propositions (logically similar to conditional propositions) to be convertible they must be biconditional or syntactically equivalent (All S are P and All P are S).

It is the issue of the convertibility of syllogisms that brings us to the second meaning of analysis and its relationship to the issue of virtue and its teachability. In mathematical demonstrations there are two distinct kinds of syllogism - analytic and synthetic. Synthetic syllogisms move deductively, or with necessity, from the more knowable universal towards the less knowable phenomena. Since they begin with some universal principle as their major premise, there is no "gain" in knowledge through such syllogisms.

Analytic syllogisms work in the reverse direction, from the less known, particular phenomenon, towards some universal, ruling principle. In mathematics such analyses are both deductive and ampliative. A "scientific demonstration" in mathematics involves the development of an analytic syllogism that is then converted into a synthetic syllogism.Mathematical analyses are "always or mostly" convertible, and that is why they may be translated directly into synthetic deductions (Aristotle, *Post.Anal.*78a5). Philosophical analyses, on the other hand, because a true conclusion can be deduced from false premises, are most often not convertible. Aristotle held that the advantage in the conversion of mathematical analyses has to do with the fact that mathematicians take as their premises, not accidental attributes, but definitions.

Socrates' concern for the way in which Meno is examining the question of the teachability of virtue can be made clearer if we rewrite our conditional arguments into categorical syllogisms:

Meno's Syllogism -
Knowledge is teachable
<u>Virtue is knowledge</u>
Virtue is teachable [hypothesis]

It is clear from this syllogism that the conditional that Meno seems to prefer - if virtue is knowledge, then it is teachable - will not satisfy the conditions of the analytical method. In Meno's syllogism we are assuming that virtue is knowledge in order to prove that it is teachable. In the analytical method the reverse is called for: "Now analysis is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis."[2] The analysis that we are to examine is the nature of a proof procedure for attempting to demonstrate that which we first have to assume.

We are again led back to the other horn of the Contrarial Pair. When we translate this other conditional - if virtue is teachable then it is knowledge - into its syllogistic form, we finally attain the analytical proof that is called for:

Socrates' Analytic Syllogism -
The teachable is knowledge
<u>Virtue is teachable [hypothesis]</u>
Virtue is knowledge

---

[2] Pappus, "Treasury of Analysis," *Greek Mathematical Works* (Cambridge, 1993), p.597.

In this syllogism we are correctly assuming the hypothesis that we want to verify - that virtue is teachable. If it leads to something not true, it is incorrect: "If someone then attacked your hypothesis itself, you would ignore him and would not answer until you had examined whether the consequences that follow from it agree with one another or contradict one another (*Phaedo* 101d)." If, however, we can confidently assert the truth of the conclusion, and the major premise is convertible, we can proceed to convert the entire syllogism to the demonstration of our hypothesis, Meno's syllogism above.

We should anticipate, however, some difficulty with the convertibility of knowledge and teachability. The inconsistency of our two arguments on the teachability of virtue hints at some equivocation in the use of these three terms. In order to find the relationship between knowledge and teachability, it is likely that we will have to seek for some "middle" that mediates between them.

If we are to understand how Plato wishes us to utilize geometrical analysis to unravel this conceptual equivocation, the math problem must help us shed some light on the nature of the convertibility of concepts.

The Mathematical Background

The second problem of the *Meno* (86d-87c) is a particular kind of analysis known as a "locus" problem. We have also noted that this particular locus is a *diorism*. Proclus recounts that one of Plato's students, Leon, first developed *diorism* as a formal method.

Proclus, in his *Commentary on the First Book of Euclid's Elements,* informs us about the distinctively philosophic interpretation of locus theorems. First, he informs us that Chrysippus "likened theorems of this sort to the (Platonic) Ideas."[3] Different than the triangular construction problems of the first half of Book I, locus constructions are determined by " a position of a line or surface producing one and the same property."[4] As such these constructions can be viewed as paradigmatic of the activity of participation, " For just as the Ideas embrace the generation of an indefinite number of particulars within determinate limits, so also in these theorems an indefinite number of cases are comprehended within determinate loci."[5] Proclus shows us how this locus-Idea can model such paradoxical phenomenon as an infinite number of different parallelograms that can each be applied to the same base between the same parallel lines, all with the same area. The perimeter of these figures can increase indefinitely without any increment of area: "Their equality is shown to result from this limitation; for the height of the parallels, which remains the same while an indefinite number of parallelograms can be thought of on the same base, shows all these parallelograms to be equal to one another."

This modeling of the locus problem as a manifestation of indefinite instances under the determination of a single Idea, recalls the philosophical problem of determining the definitional relationship between confused concepts by examining their respective extensional ranges[6]. The geometrical conditions that determine the range that the locus "rules", must somehow be translatable with the philosophical conditions of virtue and teachability if we are to sort out how Plato means for us to utilize this problem.

Aristotle names his work on the dialectical development of conceptual relations the *topos* and this term in its Latin form, *locus*, takes on the meaning of developing the conceptual range of a topic. *Topoi* or *loci* are precisely that method by which logicians work out the range and scope of their conceptual tools. For Aristotle and latter rhetoricians these terms remained parasitic on their origins in analytical geometry.

---

[3] Proclus, *A commentary on the First Book of Euclid's Elements*, (Princeton 1992), p. 311.
[4] Ibid. p. 310.
[5] p. 311.
[6] Kant utilizes the locus of a circle to illustrate this same property of mathematical concepts in the *Critique of Pure Reason*, (Indianapolis, 1987), p. 239-40.

The second mathematical problem of the *Meno* is a 'solid' locus problem. This means that it involves techniques and concepts that require in introduction of the third or "solid" dimension: complex curves derived from conic sections. The problem is to determine whether a certain rectilinear area 'X' car be inscribed as a triangle in a given circle of radius 'a' (86e).

As an analysis, we first assume that this can be done (specifically as an isosceles triangle[7]). We the must show that the constructed isosceles triangle is equal in area to a rectangle constructed on the diamete such that the rectangle on the remaining portion of the diameter is similar to the one first constructed. Thi characteristic is the equivalent of showing that the "half base" of the isosceles triangle is the mean proportional between the segments of the diameter (because it is the altitude of a triangle inscribed in a semicircle)(Figs. 1 A & B):

[7] Although any given triangular area can be constructed as an isosceles triangular area, the question of whether the scalene are always less than the isosceles area is not answered until after our analysis is completed - with the equilateral being shown to b the greatest triangular area.

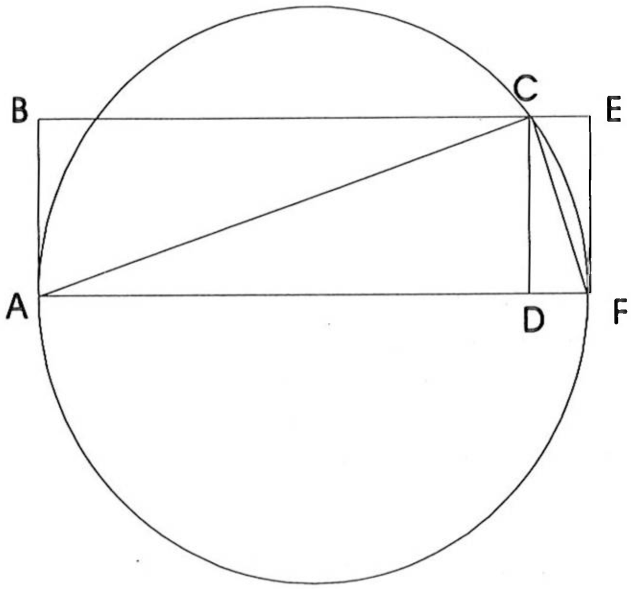
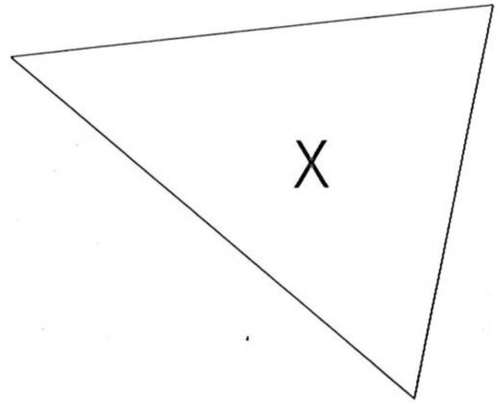


Fig. 1 A



Given Rectilinear Area

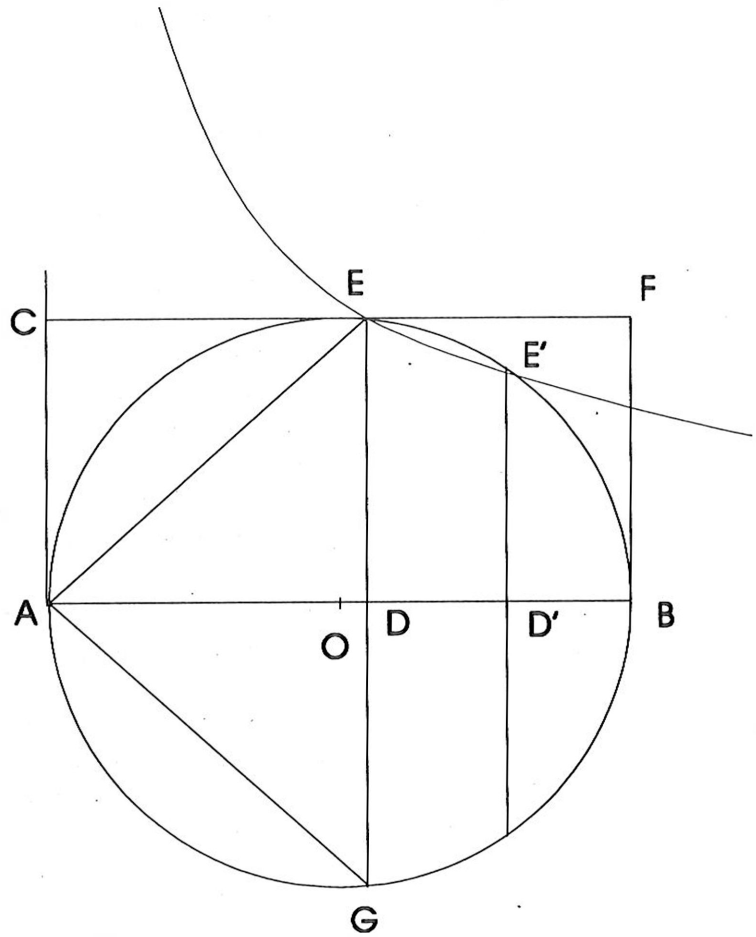
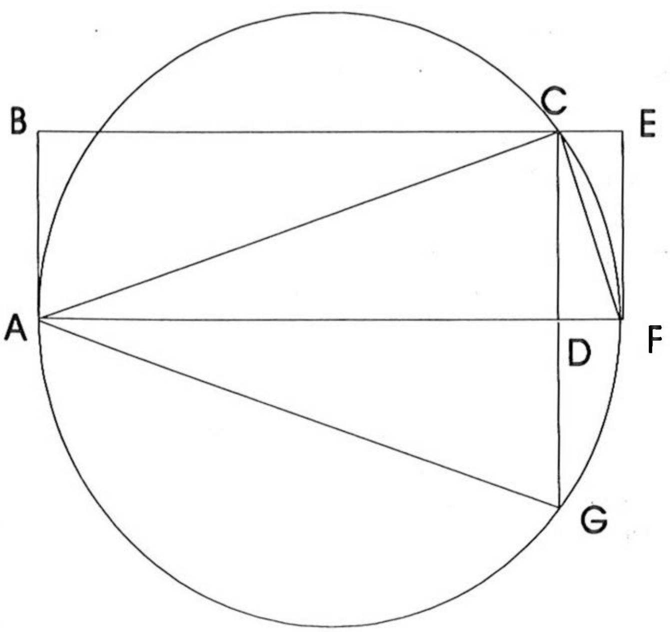


Fig. 1 C



The isosceles triangle (ACG) will always be bisected by the diameter (AF) of the circle. In other words, the two halves of the isosceles triangle can be said to be equal to a rectangle (ABCD) constructed onto that diameter.

By this construction we can see that the given condition can be satisfied by all rectangles one of whose sides lies on the given circle (AD), with one pair of opposed vertices (A, E) situated on the circumference. This condition means that angle ACF will always be a right angle and the altitude, CD, of the inscribed right triangle will always give us the required proportionality between the lines, AD:CD::CD:DF. So our original problem, that of applying an area as a triangle inscribed in a given circle, has been reduced to that of applying the original triangular area to the diameter of that given circle, in such a way that the applied rectangle (ABCD) and the rectangle constructible on the remainder of the diameter (CEFD), are similar. This similarity is given by the relationship of mean proportionality.

This is where the difficult work begins. In order to determine the areal conditions under which such mean proportionality will be possible, it is necessary to find that limiting value for which they will not attain - the maximum beyond which failure results. That is, it must be determined which isosceles triangle has the largest area. Intuitively, it is obvious that the triangle that meets this requirement is the equilateral: however, the problem is to prove the same.

The way in which this feat is accomplished is to realize that if one considers one vertex of the isosceles triangle as lying on a rectangular hyperbola ($xy = b^2$, where b^2 is the required area), and another as fixed at the origin, then the intersections of the different possible hyperbolae with the given circle ($x^2 + y^2 = 2ax$) represent those points where a second vertex of an isosceles triangle can be located to generate an inscribed isosceles triangle with area b^2, such that the first vertex, located at the origin, is such that the line connecting the two vertices (AE) indicates one of the two triangular sides of equal length.. The triangular area is maximized at the point where there is only a single solution, where the hyperbola is exactly tangent to the circle. This solution is the equilateral triangle (Figure 1 C).[8]

In the figure, triangle AEG is constructed to be equal in area to the given rectangle, $x=b^2$. The hyperbolic curve, EE′ gives all the rectangular solutions (two or one) for a given area.

To transform this analysis into a synthesis, we must exchange the minor premise (the inscription of the triangle) and the conclusion (the equating of the given figure to the Applied Rectangle) and show that the major premise is convertible. The major premise states that the "half base" of the inscribed triangle determines two similar rectangles. The converse of this condition is also true - that the similarity of the two rectangles on the diameter determines the inscribability of the equivalent triangle. So we have successfully reduced our original problem - whether a rectilinear area could be inscribed in a given circle as a triangle - to whether that same area could be constructed as a rectangle on the diameter of that same circle, such that "it is deficient by a figure similar to the very figure which is applied (87a)." It is this condition of the convertibility of the relationship between the rectangular similarity and the inscribability that allows the convertibility of the analysis into a synthesis.

As we showed earlier, this condition of the reduced construction - the similarity of the rectangles - amounts to establishing that one pair of opposed vertices (A, E) will be situated on the circumference of the circle. In this way the side of the applied rectangle in contact with the circle will always determine the vertex of an inscribed right angle, guaranteeing the required similarity, since the altitude to the hypotenuse of a right triangle is the mean proportion between the sections of the hypotenuse. This condition of a

---

[8] This *diorismic* finding of the maximum value of a hyperbolic function approximates the equivalence of modern techniques in differential calculus. Viete and Fermat "rediscovered" the procedure in a comment of Archimedes on the symmetry of finding roots for a quadratic equation. Viete had noticed in a comment of Pappus on Archimedes Proposition 61, that the constants of an equation could reveal more than just the roots, but also the "unique and the least" parts of the curve. This clue led Fermat directly into an investigation of the nature of the symmetry axes of higher curves and the relationship between tangencies and maxima-minima (Mahoney, pp 150-7).

second rectangular vertex lying on the circle may be represented as a hyperbola intersecting the circle. The hyperbola shows us that for sufficiently small rectangular areas, there are *two* points of intersection or distinct isosceles triangles. Only in the case where the hyperbolic curve is tangent to the circle will there be only a single solution, and in this case it is the maximum inscribable area - the equilateral triangle.

This problem is significant both mathematically and philosophically. The interaction between the circular and hyperbolic loci represents an equation of the fourth degree ($x^2 [2ax - x^2] = b^4$)[9]. This solution of the "cubic" equation is exactly that of another problem that Plato is reputed to have solved - the Double Cube or Delian Problem[10]. The problem was to find the length of a side of the altar at Delos, a perfect cube, if the volume was to be doubled. All of the solutions involve different ways in which to find two means between two given magnitudes. Archytas' solution specifically involves the utilization of curves in three dimensions, and is termed a "solid" locus problem. Analogously, the two problems in the *Meno* are examples of using plane and solid loci problems for finding the double of plane and solid figures. The playful possibilities evoked by the parallel between the Double Square (first problem) and the Double Cube (second problem) strongly point to Plato's sense of "measured" levity[11]. And as Socrates teasingly reminds us at the end of the dialogue, the mathematician or statesman who could resolve the problem of defining knowledge, surely "would be just like that *solid* reality among shadows (*Meno*)."[12]

There is also the interesting connection between finding two means in the Double Cube problem and, the mention of the same relationship raised latter in the *Timaeus*. There Plato asserts that between two "solids", it is necessary that there be two means as bonds (Plato, *Timaeus*, 32b). It is this condition that establishes the necessity for there being four elements. Now most people analyze this problem in the *Timaeus* as one of merely developing a continuous proportion with four terms[13]. But the problem is specifically set up as one of finding two means between two given magnitudes.

Just like finding a single mean proportion was the equivalent of solving a quadratic equation, the finding of two mean proportions was the way in which Greek mathematicians typically solved cubic equations[14]. We must now begin to wonder what significance the finding of "double middles" has for clarifying our philosophical dilemma.

Reconciling the Two Analyses

But how does Plato's locus analysis determinably guide the inquiry into the convertibility of knowledge and teachability? The convertibility of two terms makes some assumptions about the scope of their meanings. Two terms that have identical referents are extensionally convertible with each other. To the degree that the locus diorism is to bring clarity to the logic of the philosophy then it must confirm Proclus' observation about determining the definitional scope of Ideas.

It is this key insight, that of the relationship between good definitions and the nature of inclusion, whether within a "class" or a "species" or a "category", which calls up a picture of our geometrical problem. Our mathematical problem is to determine the conditions for when one kind of geometrical figure (a triangular area equal to a given rectangle) can be inscribed or included within another kind of figure (given circle). The problem involves the explicit examination of the conditions for this possibility of an "inclusion". It is this shared imagery of determining when one class or figure may be determined to "fit"

---

[8] Heath, p. 301.
[10] Ibid. p. 287.
[11] The references in the *Statesman* to the "*dunamis* of two feet" and the "*dunamis* of twice two feet" corresponds closely to this way in which mathematicians referred to finding the two means in the double cube problem. The *Statesman* is a dialogue concerned with the problem whether we could ever "duplicate" the wisdom of the original law-giver (Double Cube).
[12] Tr. Guthrie.
[13] Francis Cornford, *Plato's Cosmology* (New York, 1957), p. 47.
[14] Heath, *History*, p. 252.

into another class or figure that intimately ties these two problems, the mathematical and the philosophical, together.

My thesis is that there is a hidden logos in this dialogue. It is not the erroneous proof that virtue is not knowledge because it is unteachable. It is rather an examination of the subtleties of logical form, and the ways in which mathematical procedure can guide our philosophical precision. The dialogue holds a key to the problem of convertibility in relation to logical and geometric demonstrations.

Before one can construct a relationship between virtue, knowledge and teachability, one must be able to identify which in turn has the wider scope. This clarification of scope is the means by which the equivocation between terms may be eliminated, making definitions more precise. In examining the relationship between virtue and knowledge, Socrates showed that the scope of knowledge is seen to completely contain that of virtue, "but if there is nothing good that knowledge does not encompass, we would be right to suspect that it [virtue] is a kind of knowledge (*Meno* 87e)." This explicit reference to one idea "containing" another recalls the locus problem that is the model of the discussion. The geometrical construction is supposed to bring the same clarity to the relationship between knowledge and teachability.

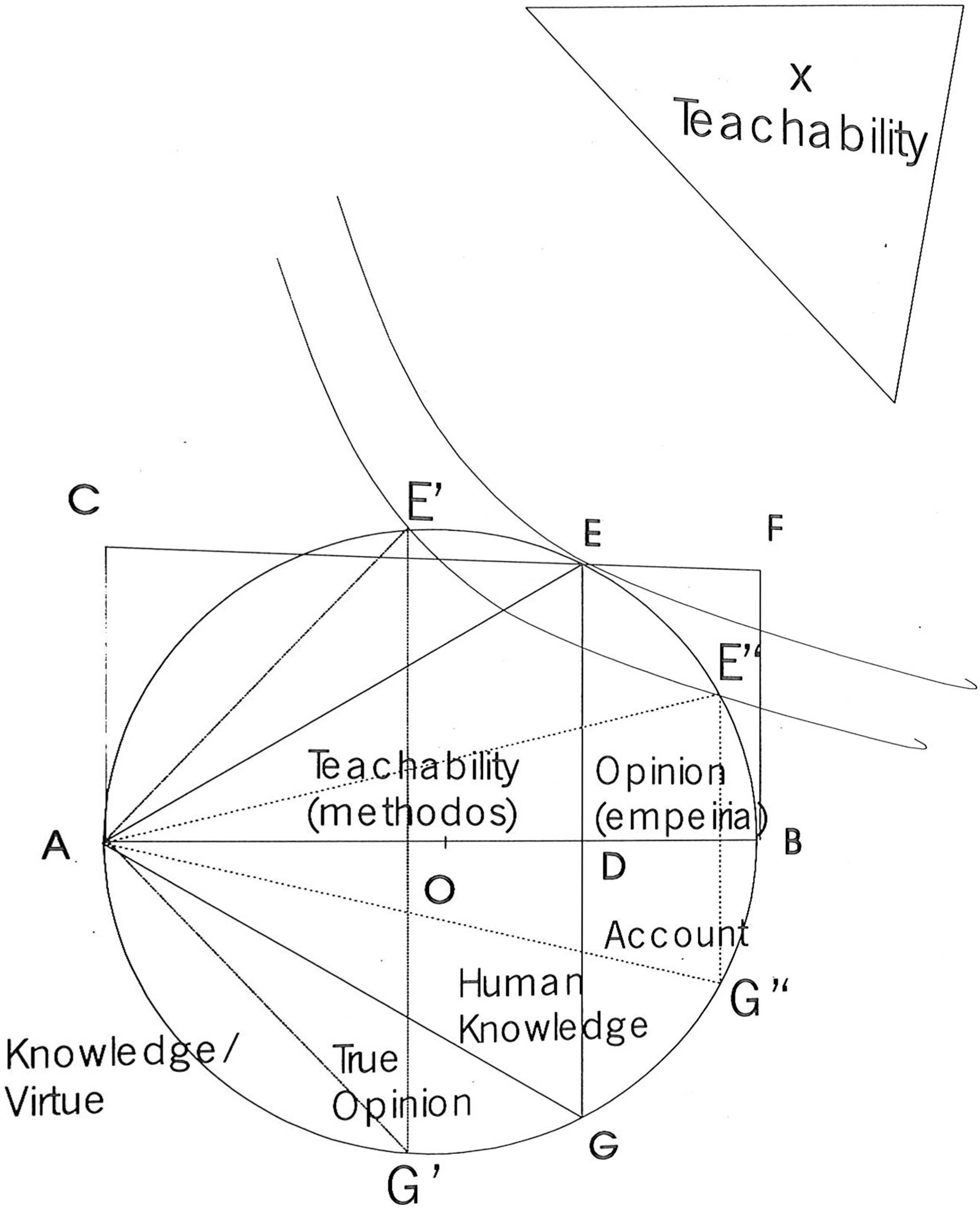
In the geometric analysis, we are to determine whether the area of a given triangular figure could be inscribed as a triangular area in a given circle. We first assumed that this could be done (as an isosceles triangle). We then showed that the constructed isosceles triangle was equal in area to a rectangle constructed on the diameter such that the rectangle on the remaining portion of the diameter was similar to the one first constructed. This characteristic is the equivalent of showing that the "half base" of the isosceles triangle is the mean proportional between the segments of the diameter (because it is the altitude of a triangle inscribed in a semicircle).

In order for this mathematical analysis to have a determinate effect on our philosophical problem, there would need to be some concrete conceptual relationship that can be reflected in our geometrical construction. In other words we must be able to identify some clear and concrete system of representation between our philosophical concepts and the figures of the geometrical construction.

Some of these correspondences seem uncontroversial. A circle completely contained by another circle represents well the relationship of inclusion (Euler Diagram). Plato certainly illuminates these kinds of logical relationships in a dialogue like the *Euthyphro*, where he asks " is the holy loved by the gods because it is holy, or is it holy because it is loved by the gods (10a)?" and "Then is all the just holy" Or is all the holy just, but not all the just holy - part of it holy, part something else (12a)?" Two congruent circles completely interposed on each other would represent identity or biconditionality between two concepts. This is the kind of relationship that he tricks Meno to agree to between knowledge and teachability.

In this representational schema there would necessarily be a distinction between curvilinear and rectilinear figures. The circular is considered most divine because it represents both a unity without parts and a perfection of balance and symmetry. Rectilinear figures, on the other hand, are archetypal of the material, the finite and the mortal. A triangle within a circle, borrowing some of the Pythagorean theology expressed in the *Timaeus* might stand for a finite concept contained within an infinite or immutable idea. This might more appropriately represent the relationship between a divine idea like knowledge and a "changeable" like opinion or teachability.

With such a form of representation we might label out mathematical construction the following way (Fig. 1 D):



X
Teachability
C
E'
E
F
E''
Teachability
(methodos)
Opinion
(empeiria)
A
B
D
O
Account
Human
Knowledge
G''
Knowledge/
Virtue
True
Opinion
G
G'

Fig. 1 D

The value of the construction in helping us determine whether knowledge and virtue are interchangeable lies "visible" before our eyes. Any triangle equal to the rectangle so constructed that the Remainder Rectangle is similar, can be "included" within the circle. But it is evident that none of these triangles is *equivalent* to the circle. The relationship of inscribability, like that between knowledge and teachability is asymmetric. The triangle may be included within the circle, *but not vice versa.* The diagram has shown us that if there is to be any convertibility between knowledge and teachability, it will not be categorical (syntactical), for there is no equivalence (biconditionality) between them. If there is to be convertibility, it can at most be conditional, as determined by some semantic relationship or "middle" between the two terms.

At this point it is appropriate to take in the full scope of our diagram and "see" which relationships can convince us of their necessary truth, as the diagonal of the square finally transfixed the slave boy's belief.

First we can note how the mathematical diagram in fact reconciles our two senses of analysis. The seeking of the convertibility of knowledge and teachability has been exactly replaced with the finding of a "middle", or mean proportion in this case. The condition for the inscribability of the translated triangle is exactly that the height of the Applied Rectangle (ED) be the mean proportion between the segments of the diameter. But this same picture also betrays the problem with such a mean. For each hyperbola intersects the circle at two points (E'D'/E"D"), determining not one mean, but two "middles." As in the *Timaeus*, the distance between knowledge and teachability is apparently of a "solid" degree.

This double middle is of course well established in the dialogue. Between knowledge (recollective teachability for Socrates) and opinion (teachability for Meno), there are two middles of true opinion and a reasoned account. But as long as these middles are separate, neither can be an adequate bridge to knowledge. Here again our diagram comes to our aid. It is precisely at that "limit" point where the hyperbola is exactly tangent to our circle that the two means converge into a single point (E). Somehow appropriately, the isosceles triangle determined by that convergent point is the one that signifies the maximum area for a triangle that can be inscribed in that circle - the "divine" equilateral triangle.

From our construction, we may conclude that while knowledge, as the circular, has a kind of Divine Nature, human knowledge, and along with it, virtue "within the human soul" is more of the nature of a mortal activity, as portrayed by the perfectly balanced equilateral triangle. The outer circle represents the Divine Ideas - Knowledge and the Good. The equilateral triangle represents the optimum capacity of human action. Its rectilinear nature makes it appropriate for representing the human condition of becoming. Its paradigmatic significance as optimum makes it ideal for signifying the instantiation of the Ideas in human knowing and virtuous action. Only an "ideal" with some kinship to the perfect could possibly represent the Divine as captured within the human[15].

It is interesting to note exactly how the equilateral triangle, as representing the best of human possibility, can be said to "partake" of the divine (the circle). For one, the triangle is "limited" by the circle. This one condition of the relationship between Form and participant refers back to the example of form as "shape" or "boundary". Socrates' first example of what a form might be was that of the limiting nature of the shape of a solid (*Meno* 76a). But the equilateral triangle may also be said to "imitate" the circle. It is the only triangle for which all of the parts are equal, and it has optimum symmetry. Although the circle does not have parts in the traditional sense, there is a way in which it can be understood as an equilateral polygon with an unlimited number of sides (Eudoxus)[16].

---

[15] This interpretation of the application of the geometric construction to the philosophical relationships is strongly confirmed in another dialogue. In considering whether the man who understands justice itself will be able to recognize the more profane version, Socrates ponders whether he be, "sufficiently versed in science if he knows the definition of the circle and of the divine sphere itself" but cannot recognize that justice which is human (*Philebus 62b*). This geometry of the *Meno* has something to do with determining the relationship between divine and "mixed" knowledge.
[16] Heath, p. 327.

On closer perusal of our figure we begin to sort out the difficulties of our conceptual relationships. Knowledge will clearly not "fit" within the scope of the teachable, but we also recognize that the converse proposition, that all the teachable was knowledge, is itself problematic. What can we conclude about the "remainder" rectangle? As a rectilinear figure it fits our condition that it is in some sense teachable, but as "outside" the conditions of our construction, it cannot be counted as in any way "knowable". So there is apparently not only knowledge that is unteachable -Divine Knowledge, but also that which is teachable that is not knowledge - mere practice or artless technique.

We are forced to recognize at this point the double irony that Meno and Socrates have each been discussing two distinct and incompatible ideas of the "teachable". For Meno, as a student of Gorgias, knowledge and learning/teaching just are the memorizing of the parts of rhetoric. There is no difference for Meno between the teachable and knowledge. They are both just a matter of learning the pieces or parts of something like virtue. They are the artless practice or *empeiria.*[17]

For Socrates, the teachable and knowledge are both of the immutable forms. Teachability, as the method of prompting recollection, is represented by the rectangular figure to be inscribed on the diameter. There are two conditions for the teachable to be truly *methodos*. First it must be a techne. Human technes, as finite activities with distinct steps, can be well pictured by the given rectilinear figure (triangle). Second, to be "knowable" a true art or techne must also be "limited" or unified by some Good or end. This is the role of capturing or limiting the triangle within the circle.

This recognition of the difference between that which is teachable with method, and that which is teachable by "mindless practice", or *empeiria*, helps focus our interpretation of the construction. The "remainder" rectangle is that teaching that is outside knowledge, and therefore, "unknowable".

But is it thoroughly unknowable? The Applied Rectangle (ACED) represents that knowledge which is directly teachable as *methodos*. The Remainder Rectangle (DEFB) would then represent that knowledge which is beyond the directly teachable. Does this make it beyond human knowing? By its determinate similarity to the rectangle of teachable knowledge, it seems to defy the label of complete unteachability. Somehow we have a determinate relationship for this "unteachable" by analogy or similarity with that which was directly teachable. That which is beyond human knowing, as not being directly teachable, is yet accessible indirectly by a relationship: "we were saying that people get the idea of what is likely through its similarity to the truth. And we just explained that in every case the person who knows the truth knows best how to determine similarities (*Phaedrus273d)*." So while empirical learning may not be knowledge proper, it may yet be proportionately related to knowledge. It is not the true, but only the "likely" or probable (*273d).* Our ignorance is somehow "measurable".

In seeking for a middle term between Divine Knowledge and teachability, in order to be able to then convert the two terms, we have fallen upon a pair of "middles." Each of these complementary middles, true opinion and a reasoned account, stands in turn as a mean between a paired set of Applied (*methodos*) and Remainder (*empeiria*) rectangles. What our diagram has convinced us of is that that human knowledge, which may be convertible with Divine Knowledge, is the convergence between true opinion and a reasoned account. This convergent middle is that represented by the paradigm of the equilateral triangle, the largest triangle inscribable within the given circle.

Conclusion

It has been my contention that Plato utilizes these geometrical constructions to parse and precise the conceptual definitions and relationships of the philosophical conversation. In order to reconcile the contradictory syllogisms for the teachability of virtue it was necessary to eliminate the equivocation within the concepts of knowledge and teachability. We had to find some middle term(s) that would bridge the equivocation, and allow us to eventually convert teachability and knowledge.

---

[17] There is a longer account of the nature of this "artless technique" in the *Phaedrus 265d-274c.*

The fact that the construction led us to the convergent limit of two means into a single, optimal paradigm, confirms that such a paradigm, as ideal human knowledge would be convertible with both teachability and Divine Knowledge.

Such a convertibility, however, would not be categorical. Only an identity, or biconditional comports that level of convertibility. Rather, there is a *conditional* convertibility of human knowledge with both teachability and Divine Knowledge and the conditions are those specifically laid out by the construction: The convergence of the double means into an optimizing paradigm.

The role of *paradigm*, or ideal example, in the Platonic theory of knowledge is illustrated in an interesting way by the special isosceles triangle, the equilateral. *Diorisms* are fundamentally involved with *solid* locus problems. These are problems that are resolved by the determining of a limit - a maximum or a minimum. The equilateral triangle represents the limit of the relationship between the hyperbolic function (area of rectangle) and the circular function.

It is through this characteristic as a maximum that the paradigm triangle may help us understand how this example of analysis overcomes what Aristotle cites as the major fault in attempting to convert analyses[18] In examining the fault within most attempted conversions, Aristotle notes:

> Some thinkers draw conclusions that do not follow syllogistically because they take [as a middle term] that which belongs to both [the major and the minor terms]. For example, this is what Caeneus does when he concludes that fire increases in a multiple proportion; for, according to him, both fire and such proportions increase fast. Now [if the premises are stated] in this manner, no syllogism is possible, but [there is a syllogism] if the fastest proportion is a multiple proportion and if fire increases according to the fastest proportion (*Post. Anal.* 78a, 1-5).

The reason why true opinion with a reasoned account is convertible with knowledge is that it is demonstrated to be the maximal possibility for knowledge within the human soul.

The optimal status of the equilateral triangle also makes it the "middle" or liminal case that defeats the skeptical claim against the possibility of knowledge. It is, in some sense, neither fully "within" nor "outside" the containing circle, as it represents the single point of tangency between the hyperbolic and circular functions[19]. It shows that there must be a *continuum* between the completely knowable (totally within the locus) and the completely unknowable (totally outside the locus). And *qualitatively* the equilateral triangle is an optimum imitation of the Divine itself. It is perfect within its constraints as a rectilinear figure. It uniquely occupies that space between the Divine and the mortal. And perfect images of reality are *conditionally* convertible with that reality. The measurable conditions of that perfection being the both the limits and the possibilities of our knowledge, as well as our virtue.

Equally this resolution answers the contemporary doubts of Gettier. The true opinion and reasoned account cannot be disassociated. For knowledge to be convertible with teachability, the two axes of knowledge, true opinion and its reasoned account, remain in reciprocal proportion to each other as they converge at the paradigm.

What can we finally say that Plato has definitively shown us with his exhibition of analytic geometry? First, he has reconfirmed the epistemological thesis of the dialogue by making visible before our eyes the solution merely discussed in the conversation. We are enabled to see the convergence of the two partial means (true opinion and reasonable account) into the limiting point that is the boundary of continuity between total knowing and total ignorance. We can see not only that there must be such continuity, but also are shown how to achieve it.

---

[18] If a true conclusion could not be proved from false [remises], analysis would be easy, for then [true premises and true conclusions] would of necessity be convertible (*Post. Anal.* 78a,10).

[19] This would be the case is being completely "within" the circle was defined as having "two" points of contact between the hyperbolic and circular loci.

Second, we are simultaneously shown the conditions for the possibility of converting an analysis into a synthesis. This is accomplished by extending the syntactical determination of the syllogism deep into the semantics grounding of our concepts.

Logical necessity is based on the spatial relationships of inclusion and exclusion, as mirrored in the interlocking circles of Euler/Venn extensional representation. If we stay solely within the logic of syntax, our syllogisms will remain the empty in the Kantian "analytical" sense, and the language we are forced to implement within those syllogisms will remain semantically ambiguous.

With geometric construction, Plato has armed us with the tools to extend the reach of syntactic rules into the realm of semantic relationships. Geometrical relationships are determinate, yet are substantially richer than the merely discrete relationship of inclusion/exclusion. The *Meno* analysis specifically demonstrates that the syntactical relationship of inclusion/exclusion can be systematically reduced to a determinate semantics of similarity and proportion. Plato has disclosed for us a geometrically determined semantics capable of setting our vague and confused concepts into a measured order - a Philosophers' Stone that can resolve base meanings into Divine ideas.

Third, we can understand that such paradigmatic relationships just are how we teach and learn the virtues. No one can have us memorize the steps to virtue. But the setting and following of virtuous action is the way home, and now Plato has given us the map to check our directional sense.

And finally, we have at least fulfilled our obligation to Plato to attempt to workout this puzzle he has dropped in our path. For clearly, without the courage to attack such intractable knots within the conversations we cannot be the heirs to any Platonic legacy:

> I do not insist that my arguments is right in all other respects, but I would contend at all costs both in word and deed as far as I could that we will be better men, braver and less idle, if we believe that one must search for the things one does not know, rather than if we believe it is not possible to find out what we do not know and that we must not look for it (*Meno* 86b).

REFERENCES CONSULTED

Albinus

1991 *The Platonic Doctrines of Albinus*, Grand Rapids: Phanes.

Alcinous

1993 *The Handbook of Platonism*, tr. John Dillon, Oxford: Clarendon.

Aristotle

1941 *The Basic Works of Aristotle*, ed., Richard McKeon, New York: Random House.

1989 *Prior Analytics*, tr. Robin Smith, Indianapolis: Hackett.

1993 *Posterior Analytics*, tr. Jonathan Barnes, Oxford: Clarendon.

Bigger, Charles

1968 *Participation: A Platonic Inquiry*, Baton Rouge: Louisiana State U. Pr.

Boyer, Carl

1956 *History of Analytic Geometry*, New York: Scripta Mathematica.

Brumbaugh, Robert

1934 *Plato's Mathematical Imagination,* Bloomington: Indiana U. Pr.

Byrne, Patrick

1997 *Analysis and Science in Aristotle,* Albany: SUNY Pr.

Cherniss, Harold

1951 "Plato as Mathematician", *Review of Metaphysics* 4: 395-425.

Coolidge, Julian Lowell

1963 *A History of Greek Geometrical Methods*, New York: Dover.

Cornford, Francis

1932 "Mathematics and Dialectic in the *Republic* VI. - VII." *Mind* XLI: 43-50.

1957 *Plato's Cosmology*, New York: Liberal Arts Pr.
Dantzig, Tobias
1955 *The Bequest of the Greeks*, New York: Charles Scribner's Sons.
Dorter Kenneth
1976 "The Dialectic of Plato's Method of Hypothesis", *Philosophical Forum*, 7:159-187.
1994 *Form and Good in Plato's Eleatic Dialogues*, Berkeley: U. Of Cal. Pr.
Euclid
1981 *The Thirteen Books of Euclid's Elements, Vols. I & II*, tr. T.L. Heath, New York: Dover.
Fowler, D. H.
1987 *The Mathematics of Plato's Academy*, Oxford: Clarendon.
Gulley, N.
1958 Greek Geometrical Analysis, *Phronesis* 3: 1-14.
Heath, Sir Thomas
1931 *A Manual of Greek Mathematics*, Oxford: Clarendon.
1949 *Mathematics in Aristotle*, Oxford: Clarendon.
1964 *Diophantus of Alexandria: A Study in the History of Greek Algebra*, New York: Dover.
1981 *A History of Greek Mathematics*, New York: Dover.
Hintikka, Jaakko, and Unto Remes
1974 *The Method of Analysis: Its Geometrical Origins and Its General Significance*, Boston Studies in the Philosophy of Science, vol. 25, Dordrecht: Reidel.
Klein, Jacob
1965 *A Commentary on Plato's Meno*, Chapel Hill: U. of North Carolina Pr.
1968 *Greek Mathematical Thought and the Origin of Algebra*, Cambridge, MA: MIT.
Knorr, Wilbur Richard
1975 *The Evolution of the Euclidean Elements*, Dordrecht: Reidel.
1993 *The Ancient Tradition of Geometric Problems*, New York: Dover.
Lachterman, David
1989 *The Ethics of Geometry*, New York: Routledge.
Mahoney, Michael
1967 *The Royal Road*, Princeton: Princeton U. Pr.
1968 "Another Look at Greek Geometrical Analysis", *Archive for History of Exact Sciences* 5: 319-48.
1973 *The Mathematical Career of Pierre de Fermat*, Princeton: Princeton U. Pr
Maziarz, Edward and Greenwood, Thomas
1950 *Greek Mathematical Philosophy*, New York: Philosophical Library.
Mueller, Ian
1981 *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, Cambridge, MA: MIT Pr.
Otte, Michael and Panza, Marco, ed.
1997 *Analysis and Synthesis in Mathematics*, Dordrecht: Kluwar.
Plato
1961 *Collected Dialogues*, ed. Edith Hamilton and Huntington Cairns, Princeton: Princeton U. Pr.
1981 *Five Dialogues*, tr. G. M. A. Grube, Indianapolis: Hackett.
1986 *Plato's Statesman*, tr. Seth Bernardete, Chicago: University of Chicago Pr.
1990 *The Theaetetus of Plato*, tr. M. J. Levett, Indianapolis: Hackett.
1993 *Philebus*, tr. Dorothea Frede, Indianapolis: Hackett.
Sophist, tr. Nicholas White, Indianapolis: Hackett.
1996 *Parmenides*, tr. Marry Louise Gill and Paul Ryan, Indianapolis: Hackett.

Playfair, John
1822 *Works of John Playfair,* Edinburgh: A. Constable & Co.
Proclus
1970 *A Commentary on the First Book of Euclid's Elements*, Princeton: Princeton U. Pr.
Sayre, Kenneth
1969 *Plato's Analytic Method*, Chicago: U. of Chicago.
Scolnicov, Samuel
1975 "Hypothetical Method and Rationality in Plato", *Kantstudien* 66: 157-62.
Smith, David, ed.
1959 *A Source Book in Mathematics*, New York: Dover.
Stenius, Erik
1978 "Foundations of Mathematics: Ancient and Modern", *Dialectica* 32: 255-89.
Struik, D. J., ed.
1990 *A Source Book in Mathematics 1200 - 1800*, Princeton: Princeton U. Pr.
Szabo, Arpad
1978 *The Beginnings of Greek Mathematics*, Dordrecht: Reidel.
Theon of Smyrna
1979 *Mathematics Useful for Understanding Plato*, tr. Robert and Deborah Lawlor, San Diego: Wizards Bookshelf.
Thomas, Ivor
1993 *Greek Mathematics, Vols. 1 & 2*, Cambridge, MA: Harvard U. Pr.
Wedberg, Anders
1930 *Plato's Philosophy of Mathematics,* tr. G. R. Mure, Stockholm: Almquist & Wiksell.
Van der Waerden, B. L.
1963 *Science Awakening*, New York: Wiley.